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Unsteady heat conduction in two-dimensional two slab-shaped regions. Exact closed-form solution and results

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Abstract

The unsteady heat conduction analysis for multi-directional piecewise-homogeneous bodies is generally held to be complex and demanding, possibly explaining why practical guidelines for thermal field calculation are few and far between. The proposed solution method represents an extension of the new, '*natural*' analytic approach derived in companion papers for solving one-dimensional multi-layer problems of time-dependent heat conduction. As the approach is new, it is presented in full, together with the complete temperature double-series solution prepared for computer implementation. By setting thermal diffusivity ratio unitary and assuming a uniform distribution of initial temperature, it emerges that, all other things being equal, the transient thermal response can be expressed as the product of two, separated, one-directional solutions, one across the layers and the other along the composite slab. The formulation deals properly with thermal conductivity ratios of all magnitudes. An efficient and accurate procedure of computing eigenvalues is given. Graphical and numerical output is presented and discussed.

Keywords: Composites; Conduction; Unsteady

1. Introduction

Exact closed-form solutions in multi-layer multi-dimensional transient heat conduction instead of numerical commercial programs ready to hand can be successfully and efficiently used in several engineering applications such as: (1) double heat flux conductimeter (three-layer slab); (2) thermal analysis of building multi-layer walls; (3) thermal insulation for cryogenic systems; (4) solar ovens (composite cylinder placed at the focus), and (5) double layer bodies (of innovative materials) irradiated by laser sources. Additionally, an involved closed-form solution represents for the user a simplified task. The objective of this article is to provide extremely accurate solutions ready for computer implementation for verifying the accuracy of complex approximate numerical (commercial) programs for multi-dimensional multi-layer transient heat conduction problems [1]. In such a way, the comparisons with its solutions are very reliable.

The analysis of multi-directional time-dependent heat conduction in composite media consisting of several different parallel layers in contact may be analytically performed following different approaches below described.

The *orthogonal expansion technique* derived by Padovan [2], Salt [3,4], and Mikhailov and Özişik [5]. Padovan [2] analysed a 3-D configuration consisting of several distinct arbitrary thermally anisotropic subdomains with internal heat generation and in perfect thermal contact. A 3-D anisotropic generalised version of the classical Sturm–Liouville procedure [6,7] was established. However, the '*product formula*' was not applied to the 3-D space-variable function and, therefore, the final temperature solution was presented only formally. Salt [3,4] examined a 2-D multi-layer isotropic-composite slab (without internal heat source) whose layers were in perfect thermal contact. The composite plate was subjected to linear homogeneous boundary conditions in the direction perpendicular to the layers. Salt [4] demonstrated first that the eigenvalues for the

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Nomenclature

a_i	dimension of the ith layer along x (Fig. 1)
b	dimension of each layer along y (Fig. 1)
Bi_i	Biot number for the ith layer: $h_i a_1/k_1$
$Bi_{h,i}$	Biot number for the homogeneous slab associated to the two-layer slab: $h_i L/k_h$
$f_i(x,y)$	arbitrary initial temperature distribution for the ith layer
$F_i(x,y)$	arbitrary initial temperature difference for the ith layer: $T_\infty - f_i(x,y)$
h_i	heat transfer coefficient for the ith layer at the x-boundary side of Fig. 1
h_l	heat transfer coefficient for the slab at the y-boundary side of Fig. 1
k_h	thermal conductivity of the homogeneous slab associated to the two-layer slab
k_i	thermal conductivity of the ith layer
L	total dimension of the two-layer slab along x (Fig. 1): $a_1 + a_2$
$N_{x,mn}$	mth × nth x-norm defined by Eq. (39)
$N_{y,m}$	mth y-norm defined in Table 2
t	time
T_i	temperature for the ith layer
T_0	uniform initial temperature of the composite domain
T_∞	fluid temperature
x, y	rectangular space coordinates
$X'_{i,mn}(x)$	mth × nth x-eigenfunction corresponding to $\lambda_{ix,mn}$ for the ith layer
$Y'_m(y)$	mth y-eigenfunction related to $\lambda_{y,m}$ for each of the two layers

Greek symbols

α_i	thermal diffusivity of the ith layer
$\beta_{\xi,n}$	nth dimensionless eigenvalue for ξ-direction (when $\alpha_1 = \alpha_2$): $\lambda_{x,n} a_1$
$\beta_{\xi,h,n}$	nth dimensionless transverse eigenvalue of the homogeneous slab associated to the two-layer slab (when $\alpha_1 = \alpha_2$): $\lambda_{x,h,n} L$
$\beta_{\psi,m}$	mth dimensionless eigenvalue for ψ-direction: $\lambda_{y,m} a_1$
γ, μ	geometric ratios: a_2/a_1, b/a_1
$\zeta_{\xi,n}$	nth initial guess for the nth dimensionless transverse eigenvalue $\beta_{\xi,n}$
θ_i	temperature difference for the ith layer: $T_\infty - T_i$
θ_0	uniform initial temperature difference: $T_\infty - T_0$
Θ_i	dimensionless temperature for the ith layer: θ_i/θ_0
κ	thermal conductivity ratio: k_2/k_1
$\lambda_{ix,mn}$	mth × nth eigenvalue of the ith layer for x-direction
$\lambda_{x,n}$	nth eigenvalue for x-direction (when $\alpha_1 = \alpha_2$)
$\lambda_{y,m}$	mth eigenvalue along y
ξ, ψ	dimensionless rectangular space coordinates: x/a_1, y/a_1
τ	dimensionless time (when $\alpha_1 = \alpha_2 = \alpha$): $\alpha t/a_1^2$

Subscripts

i	index (i.e., 1 or 2)
l	index (i.e., 0 or b)
m	integer number (including zero)
n	integer number (including zero)
0, b	lower ($y = 0$) and upper ($y = b$) sides of the rectangular in Fig. 1
1	first layer ($-a_1 \leqslant x \leqslant 0; 0 \leqslant y \leqslant b$); left side ($x = -a_1$) of the domain in Fig. 1
2	second layer ($0 \leqslant x \leqslant a_2; 0 \leqslant y \leqslant b$); right side ($x = a_2$) of the domain in Fig. 1

solution transverse to the layers can progressively become imaginary in the region with higher thermal diffusivity. However, no numerical example was presented. Mikhailov and Özişik [5], instead, analysed the 3-D version of Salt's problem. The eigenproblem associated to the solution of the transient multi-layer heat conduction under consideration was split up into three, separated, 1-D eigenproblems. In particular, through a very clever algebraic setting, the eigenvalue problem in the direction perpendicular to the layers coincided with the classical 1-D Sturm–Liouville procedure. However, no numerical result was provided and no consideration concerning imaginary transverse eigenvalues was given.

The *Green's function solution method* developed by Beck et al. [8,9]. In [8] the method was applied to a 3-D two-layer (firmly joined) isotropic-composite slab with internal heat source. Numerical values for three examples were presented but the solution transverse to the layers did not include the contribution of possible imaginary eigenvalues. In [9], instead, the technique was applied to a 3-D two-layer orthotropic-composite slab with internal energy generation and contact thermal resistance between the layers. Numerical values for three examples were given and the complete solution retained all eigenvalues, real and imaginary [10,11], in accordance with Salt's result [4]. Additionally, in both Refs. [10,11], the concept of 'time partitioning' [12,13] was used in order to achieve much more efficient temperature solutions (although a tiny bit less accurate).

The *Laplace transform approach* used by Levine [14], and Kozlov and Mandrix [15,16]. Levine [14] considered a spherical surface subjected to a homogeneous boundary condition of the first kind and surrounded by a composite medium made up of two different isotropic concentric regions (the outer one was assumed infinitely large) where a point heat source was located. The problem was considered two dimensional and no numerical application was presented. Kozlov and Mandrix [15,16], instead, examined a 2-D isotropic-composite system (without internal heat source) consisting of a bounded cylinder and a semibounded body that perfectly (in a thermal sense) touch each other at one of the end surfaces of the cylinder. The composite configuration was subjected to homogeneous boundary conditions in the direction perpendicular to the layers. However, the solution was obtained only in the region of Laplace transforms and, therefore, no numerical result was provided.

All the research workers mentioned above agree the solution is able to deal only with homogeneous boundary conditions of the first kind and second kind in the direction parallel to the layers, since the (linear) homogeneous boundary condition of the third kind unconditionally produces mathematical incompatibilities.

Recently, a '*natural*' analytic approach for solving one-dimensional transient heat conduction in multilayer composite media was derived by de Monte [17,18]. The main feature of this approach is to retain the thermal diffusivity of each layer on the side of the heat conduction equation (modified from the application of the separation-of-variables method) where the time-dependent function is collected. Making this, the modified heat conduction equation by itself represents a transparent statement of the physical phenomena involved and straightly leads to a series solution of the problem where the time-variable function is explicitly dependent on the thermal diffusivity.

In the present paper the new, natural analytic solution method is extended to two-dimensional composites of two rectangular parallel layers which are in perfect thermal contact. By assuming an appropriate separation of variables, the eigenvalue problem associated to the general temperature solution is split up into two one-dimensional eigenproblems. The eigenvalue problem in the direction of the layers is a special case of a more general eigenproblem called the Sturm–Liouville problem [6,7]. The eigenproblem in the direction perpendicular to the layers, instead, is characterised by real and imaginary eigenvalues and, therefore, does not lead to the class of Sturm–Liouville eigenproblems. In particular, the eigenfunctions across the layers (corresponding to the transverse eigenvalues) are chosen with the target to identify and stress '*those algebraic terms*' which account for the heat conduction in the direction parallel to the layers and affect the thermal field in the direction where the slab is two-layered (clearly, no similar term exists in a two-dimensional homogeneous domain). Therefore, a new orthogonality property (embodying the transverse norm) is developed by the author and then used as a straightforward matter to achieve the final double-series form of the exact closed-form solution.

A numerical example is presented and discussed at the end of the paper to show how the proposed technique works. In particular, the transverse eigenvalues are computed by the use of an efficient and accurate procedure developed by the author and fully treated in Ref. [19]. It is based on the concept of '*homogeneous* (*single-layer*) *rectangular domain*' equivalent to the two-dimensional two-layered domain here under discussion. This concept is used for searching the initial guesses for the transverse eigenvalues of the eigencondition associated to the transient multi-layer problem. Then, the Müller root-finding iteration [20] is used to compute the eigenvalues. Furthermore, by setting thermal diffusivity ratio unitary and assuming a uniform distribution of initial temperature, the isothermal curves within the 2-D two-layer slab during the transient heat transfer between slab and surrounding fluid are plotted at different times.

2. Formulation of the problem

Consider a two-layer rectangle with edges which run parallel with the x and y coordinate directions respectively, as shown in Fig. 1. Initially $(t = 0)$ the two-dimensional isotropic composite slab is at a specified temperature $f(x, y)$. Then, for $t > 0$, the surface of the slab is brought into contact with a fluid whose temperature T_∞ is constant with time. The heat transfer coefficient is assumed to be independent of time and temperature but different on all sides of the plate. Furthermore, there is no internal heat source and the material thermal properties are temperature independent and uniform within each layer. Under these assumptions, the heat conduction differential equation and the outer and inner boundary conditions are:

$$\frac{\partial^2\theta_i}{\partial x^2} + \frac{\partial^2\theta_i}{\partial y^2} = \frac{1}{\alpha_i}\frac{\partial\theta_i}{\partial t} \tag{1}$$

$$\mp k_i\left(\frac{\partial\theta_i}{\partial x}\right)_{x=\mp a_i} + h_i\theta_i(x = \mp a_i, y, t) = 0 \tag{2}$$

$$-k_i\left(\frac{\partial\theta_i}{\partial y}\right)_{y=0} + h_0\theta_i(x, y = 0, t) = 0 \tag{3}$$

$$k_i\left(\frac{\partial\theta_i}{\partial y}\right)_{y=b} + h_b\theta_i(x, y = b, t) = 0 \tag{4}$$

$$\theta_1(x = 0, y, t) = \theta_2(x = 0, y, t) \tag{5}$$

$$k_1\left(\frac{\partial\theta_1}{\partial x}\right)_{x=0} = k_2\left(\frac{\partial\theta_2}{\partial x}\right)_{x=0} \tag{6}$$

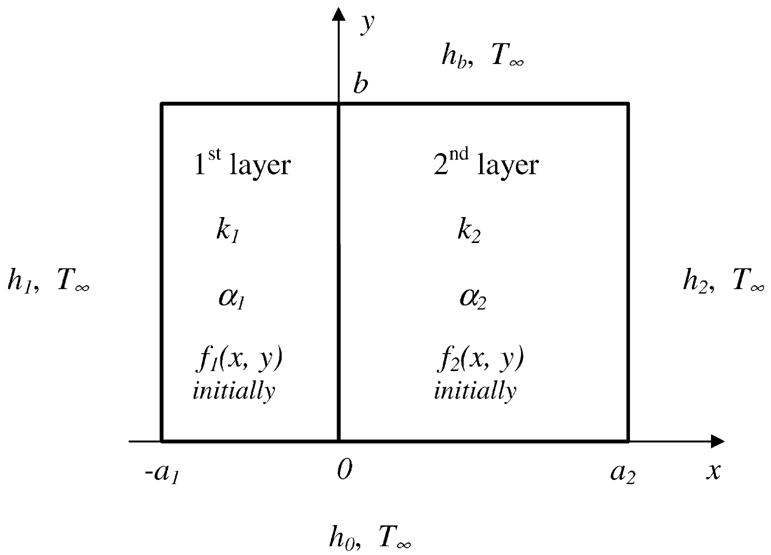

Fig. 1. Boundary and initial conditions for heat conduction analysis of a two-layer rectangular region.

where the negative sign in Eq. (2) is valid for $i = 1$, while the positive sign for $i = 2$. In Eq. (2) the heat transfer coefficient h_i may be chosen in such a way as to yield either Dirichlet, Neumann or Cauchy type boundary conditions. Eq. (5) is only valid if no thermal contact resistance occurs at the separation points of the two layers (continuity of temperature) [21]. For sake of completeness, it should be said that Eq. (6) is only valid if there is no heat source distribution over the surface of separation of the two layers (i.e., no prescribed discontinuity in heat flux). Additionally, the initial condition

$$\theta_i(x, y, t = 0) = F_i(x, y) \tag{7}$$

is valid. Fig. 1 shows that equation $x = 0$ coincides with the line of separation of the two slab-shaped regions. This choice allows the analytical development to be notably simplified when the inner boundary conditions, i.e. Eqs. (5) and (6), are applied to search the temperature solution.

3. Separating the variables

Since the outer boundary conditions (2)–(4) are homogeneous, the product or separation formula

$$\theta_i(x, y, t) = X_i(x)Y_i(y)G_i(t) \tag{8}$$

may be applied in order to find a solution to the differential equation (1). However, when product solution is used, the fulfilment of the inner boundary conditions (5) and (6) is only possible (as will be better shown in Section 5) when, as outer boundary conditions (3) and (4) in the y-direction, constant temperature at zero level or adiabatic edges are rigorously prescribed. The functions $X_i(x)$, $Y_i(y)$ and $G_i(t)$ satisfy the following differential equation from (1)

$$\frac{1}{X_i}\frac{d^2X_i}{dx^2} + \frac{1}{Y_i}\frac{d^2Y_i}{dy^2} = \frac{1}{\alpha_i G_i}\frac{dG_i}{dt} \tag{9}$$

It may be noted that, in separating the variables, the thermal diffusivity α_i is retained on the right side of Eq. (9), where the time-dependent function $G_i(t)$ is collected, according to the natural analytic approach developed by de Monte [17,18]. The product formula (8) yields three easy to solve ordinary differential equations from (9)

$$\frac{d^2X_i}{dx^2} + \lambda_{ix}^2 X_i = 0 \tag{10}$$

$$\frac{d^2Y_i}{dy^2} + \lambda_{iy}^2 Y_i = 0 \tag{11}$$

$$\frac{dG_i}{dt} + (\lambda_{ix}^2 + \lambda_{iy}^2)\alpha_i G_i = 0 \tag{12}$$

The space-variable functions $X_i(x)$ and $Y_i(y)$ satisfy the differential equations (10) and (11), respectively, known as the differential equations governing harmonic oscillations. The general solutions are

$$X_i(x) = A_{ix}\cos(\lambda_{ix}x) + B_{ix}\sin(\lambda_{ix}x) \tag{13}$$

$$Y_i(y) = A_{iy}\cos(\lambda_{iy}y) + B_{iy}\sin(\lambda_{iy}y) \tag{14}$$

where A_{ix}, A_{iy}, B_{ix}, B_{iy} $(i = 1, 2)$ are the *integration constants* along x and y related to the first and second layers. Instead, the solution of the differential equation

(12) for the time function $G_i(t)$ is the decay exponential function

$$G_i(t) = e^{-(\lambda_{ix}^2+\lambda_{iy}^2)\alpha_i t} \quad (15)$$

4. Application of outer boundary conditions

It follows from the outer boundary conditions (2)–(4) that

$$A_{ix} = \pm B_{ix} \Pi_{ix}(\lambda_{ix}) \quad (16)$$

$$\Pi_{ix}(\lambda_{ix}) = \frac{k_i \lambda_{ix} + h_i \tan(\lambda_{ix} a_i)}{h_i - k_i \lambda_{ix} \tan(\lambda_{ix} a_i)} \quad (17)$$

$$A_{iy} = B_{iy} \Pi_{iy}(\lambda_{iy}) \quad (18)$$

$$\Pi_{iy}(\lambda_{iy}) = \frac{k_i \lambda_{iy}}{h_0} \quad (19)$$

$$\tan(\lambda_{iy} b) = \frac{k_i \lambda_{iy}(h_0 + h_b)}{(k_i \lambda_{iy})^2 - h_0 h_b} \quad (20)$$

In Eq. (16) the positive sign is valid when $i = 1$ (first layer), while the negative sign when $i = 2$ (second layer). In view of Eq. (18), the function $Y_i(y)$ given by Eq. (14) becomes

$$Y_i(y) = B_{iy}[\sin(\lambda_{iy} y) + \Pi_{iy}(\lambda_{iy}) \cos(\lambda_{iy} y)] = B_{iy} Y_i'(y) \quad (21)$$

where λ_{iy} is any positive root other than zero of the transcendental equation (20).

5. Application of inner boundary conditions

It follows from the inner boundary conditions (5) and (6) (compatibility conditions) that

$$B_{1x} \Pi_{1x} Y_1(y) e^{-(\lambda_{1x}^2+\lambda_{1y}^2)\alpha_1 t} = -B_{2x} \Pi_{2x} Y_2(y) e^{-(\lambda_{2x}^2+\lambda_{2y}^2)\alpha_2 t} \quad (22)$$

$$k_1 B_{1x} \lambda_{1x} Y_1(y) e^{-(\lambda_{1x}^2+\lambda_{1y}^2)\alpha_1 t} = k_2 B_{2x} \lambda_{2x} Y_2(y) e^{-(\lambda_{2x}^2+\lambda_{2y}^2)\alpha_2 t} \quad (23)$$

Since conductivity and diffusivity are discontinuous at the interface of the two layers, i.e. at $(x = 0, y)$ where $y \in [0, b]$, Eqs. (22) and (23) are *only* verified when:

$$Y_1(y) = Y_2(y) \quad (24)$$

$$(\lambda_{1x}^2 + \lambda_{1y}^2)\alpha_1 = (\lambda_{2x}^2 + \lambda_{2y}^2)\alpha_2 \quad (25)$$

$$B_{2x} = -B_{1x} \frac{\Pi_{1x}}{\Pi_{2x}} \quad (26)$$

$$B_{2x} = B_{1x} \frac{k_1 \lambda_{1x}}{k_2 \lambda_{2x}} \quad (27)$$

Bearing in mind Eq. (21), it should be noted that Eq. (24) is only verified when $B_{1y} = B_{2y} = B_y$ and the heat transfer coefficients h_0 and h_b, appearing in the y-boundary conditions (3) and (4), respectively, assume the limit values 0 (boundary kept insulated, i.e. homogeneous boundary condition of the second kind) and ∞ (boundary kept at zero temperature, i.e. homogeneous boundary condition of the first kind), as shown in Table 1. These values, in fact, lead to four different boundary-value problems along y characterised by

- the same undetermined parameter along y, i.e. $\lambda_{1y} = \lambda_{2y} = \lambda_y$, where λ_y is any positive root of Eq. (20) represented in a simplified form in Table 1;
- the same y-position function in each of the two layers, i.e. $Y_1'(y) = Y_2'(y) = Y'(y)$ which is clearly independent both of the thermal conductivity of each region and of the heat transfer coefficients h_0 and h_b (see Table 1).

Therefore, in this paper, the analytic treatment will account for only the four cases of y-boundary conditions given in Table 1.

In view of Eq. (16), the function $X_1(x)$ given by Eq. (13) for $i = 1$ becomes

$$X_1(x) = B_{1x}[\sin(\lambda_{1x} x) + \Pi_{1x}(\lambda_{1x}) \cos(\lambda_{1x} x)] \quad (28)$$

Similarly, in view of Eqs. (16) and (27), the function $X_2(x)$ defined by Eq. (13) for $i = 2$ reduces to

$$X_2(x) = B_{1x} \frac{k_1 \lambda_{1x}}{k_2 \lambda_{2x}} [\sin(\lambda_{2x} x) - \Pi_{2x}(\lambda_{2x}) \cos(\lambda_{2x} x)] \quad (29)$$

Furthermore, comparing Eqs. (26) and (27) yields the following result

$$\Pi_{1x}(\lambda_{1x}) = -\frac{k_1 \lambda_{1x}}{k_2 \lambda_{2x}} \Pi_{2x}(\lambda_{2x}) \quad (30)$$

Table 1
Homogeneous boundary conditions in the direction parallel to the layers

h_0	h_b	Case	Eq. (3)	Eq. (4)	Eq. (19)	Eq. (20)	Eq. (21)
∞	∞	1	$\theta_i(x, y = 0, t) = 0$	$\theta_i(x, y = b, t) = 0$	$\Pi_{iy}(\lambda_{iy}) = 0$	$\sin(\lambda_{iy} b) = 0$	$Y_i'(y) = \sin(\lambda_{iy} y)$
0	0	2	$(\partial\theta_i/\partial y)_{y=0} = 0$	$(\partial\theta_i/\partial y)_{y=b} = 0$	$\Pi_{iy}(\lambda_{iy}) = \infty$	$\sin(\lambda_{iy} b) = 0$	$Y_i'(y) = \cos(\lambda_{iy} y)$
∞	0	3	$\theta_i(x, y = 0, t) = 0$	$(\partial\theta_i/\partial y)_{y=b} = 0$	$\Pi_{iy}(\lambda_{iy}) = 0$	$\cos(\lambda_{iy} b) = 0$	$Y_i'(y) = \sin(\lambda_{iy} y)$
0	∞	4	$(\partial\theta_i/\partial y)_{y=0} = 0$	$\theta_i(x, y = b, t) = 0$	$\Pi_{iy}(\lambda_{iy}) = \infty$	$\cos(\lambda_{iy} b) = 0$	$Y_i'(y) = \cos(\lambda_{iy} y)$

where λ_{2x} may be evaluated by means of Eq. (25), where $\lambda_{1y} = \lambda_{2y} = \lambda_y$, as

$$\lambda_{2x} = \left[\left(\frac{\alpha_1}{\alpha_2}\right)\lambda_{1x}^2 + \left(\frac{\alpha_1}{\alpha_2} - 1\right)\lambda_y^2\right]^{1/2} \quad (31)$$

6. The eigenvalue problem in the *y*-direction

The homogeneous y-boundary value problem posed by the differential equation (11) and the y-boundary conditions (3) and (4), where θ_i should be substituted with Y_i, is a special case of a more general eigenvalue problem called the one-dimensional linear Sturm–Liouville boundary value eigenproblem treated in [6,7] for which a series of general theorems are valid [6,7,21,22]. Concerning this, Table 2 shows that the roots (y-eigenvalues) of the simplified y-eigencondition (20) are infinite, distinct and real (the y-eigenvalues form a monotically increasing infinite series) in accordance with the Sturm–Liouville procedure: $\lambda_{y,0} < \lambda_{y,1} < \cdots < \lambda_{y,m} \ldots$ $(m = 0, 1, 2, 3, \ldots)$. Therefore, there are infinite functions having the simplified form (21) given in Table 1, each corresponding to a consecutive value of the y-eigenvalues $\lambda_{y,m}$:

$$Y_m(y) = B_{y,m} Y'_m(y) \quad y \in [0, b] \quad (32)$$

where the functions $Y'_m(y)$ are shown in Table 2. They have been assumed as the y-eigenfunctions corresponding to the y-eigenvalues $\lambda_{y,m}$, and are orthogonal functions [21,22], i.e. it holds that

$$\int_{y=0}^{b} Y'_m Y'_j \, \mathrm{d}y = \begin{cases} 0 & \text{for } m \neq j \\ N_{y,m} & \text{for } m = j \end{cases} \quad (33)$$

where $N_{y,m}$ represents the mth y-norm, which may be evaluated as indicated in Table 2.

7. The eigenvalue problem in the *x*-direction

The homogeneous x-eigenvalue problem posed by the differential equation (10) and the x-boundary conditions (2), (5) and (6), where θ_i $(i = 1, 2)$ should be replaced by X_i, does not lead to the class of Sturm–Liouville eigenproblems [6,7] when the thermal properties (density, specific heat and thermal conductivity) are stepwise functions in the x-direction [6,7]. The reason for all that is due to the choice here adopted about the parameters used in separating the variables. In other words, two undetermined parameters (λ_{ix} and λ_{iy}) have been introduced to characterise the position functions $X_i(x)$ and $Y_i(y)$, as shown in Eqs. (10) and (11), but no undetermined parameter has been introduced for the time function $G_i(t)$ (see Eq. (12)).

Consequently, substituting Eq. (31) in Eq. (30), we obtain a transcendental equation (x-eigencondition) whose roots λ_{1x} are the x-eigenvalues related to the first layer and may clearly be real or imaginary according to what was established in Refs. [4,9,10] and at variance with the Sturm–Liouville procedure [6,7]. Since they depend on $\lambda_{y,m}$ $(m = 0, 1, 2, 3, \ldots)$, there exist $m \times n$ functions $(n = 0, 1, 2, 3, \ldots)$ having the form (28), each corresponding to a consecutive value of the x-eigenvalues $\lambda_{1x,mn}$:

$$X_{1,mn}(x) = B_{1x,mn} X'_{1,mn}(x) \quad x \in [-a_1, 0] \quad (34)$$

Once the eigenvalues $\lambda_{y,m}$ and $\lambda_{1x,mn}$ have been calculated, Eq. (31) provides the x-eigenvalues $\lambda_{2x,mn}$ $(n = 0, 1, 2, 3, \ldots)$ related to the second layer, which may be real or imaginary according to what was previously said. Consequently, there are $m \times n$ functions having the form (29), each corresponding to a consecutive value of the x-eigenvalues $\lambda_{2x,mn}$:

$$X_{2,mn}(x) = B_{1x,mn} \frac{k_1 \lambda_{1x,mn}}{k_2 \lambda_{2x,mn}} X'_{2,mn}(x) \quad x \in [0, a_2] \quad (35)$$

Table 2
The eigenvalue problem in the y-coordinate direction

Case	y-Eigencondition	y-Eigenvalues $\lambda_{y,m}$ $(m = 0, 1, 2, 3, \ldots)$	y-Eigenfunctions $Y'_m(y)$ $y \in [0, b]$	y-Norm $N_{y,m}$	$c_{y,m}$ coefficients-Eq. (52) with $\Im_y(y) = 1$
1	$\sin(\lambda_y b) = 0$	$m\pi/b$[a]	$\sin(\lambda_{y,m} y)$	$b/2$	0 for $m = 2, 4, 6, \ldots$ $4/(m\pi)$ for $m = 1, 3, 5, \ldots$
2	$\sin(\lambda_y b) = 0$	$m\pi/b$	$\cos(\lambda_{y,m} y)$	$b/2$[b]	1 for $m = 0$ 0 for $m = 1, 2, 3, \ldots$
3	$\cos(\lambda_y b) = 0$	$(m + 1/2)\pi/b$	$\sin(\lambda_{y,m} y)$	$b/2$	$1/[(2m + 1)\pi]$ for $m = 0, 1, 2, 3, \ldots$
4	$\cos(\lambda_y b) = 0$	$(m + 1/2)\pi/b$	$\cos(\lambda_{y,m} y)$	$b/2$	$(-1)^m/[(2m + 1)\pi]$ for $m = 0, 1, 2, 3, \ldots$

[a] $m = 1, 2, 3, \ldots$
[b] $N_{y,0} = b$.

The dimensionless functions $X'_{1,mn}(x)$ and $X'_{2,mn}(x)$, which appear in Eqs. (34) and (35), have been assumed as the x-eigenfunctions corresponding to the x-eigenvalues $\lambda_{1x,mn}$ (first layer) and $\lambda_{2x,mn}$ (second layer), respectively, and are defined as (see Eqs. (28) and (29))

$$X'_{1,mn}(x) = \sin(\lambda_{1x,mn}x) + \Pi_{1x,mn}\cos(\lambda_{1x,mn}x) \quad x \in [-a_1, 0] \tag{36}$$

$$X'_{2,mn}(x) = \sin(\lambda_{2x,mn}x) - \Pi_{2x,mn}\cos(\lambda_{2x,mn}x) \quad x \in [0, a_2] \tag{37}$$

where $\Pi_{1x,mn} = \Pi_{1x}(\lambda_{1x,mn})$ and $\Pi_{2x,mn} = \Pi_{2x}(\lambda_{2x,mn})$. It may be proven that the x-eigenfunctions $X'_{1,mn}(x)$ and $X'_{2,mn}(x)$ are orthogonal functions. However, they do not satisfy the well-known orthogonality property given by Özişik [22] for 1-D multi-layer composite media and used by Haji-Sheikh and Beck [9] in the direction perpendicular to the layers of a 3-D two-region slab. Furthermore, they do not verify even the orthogonality relations proposed by de Monte for both 1-D two-layer slabs [17] and 1-D multi-layer bodies [18]. The reason for all that is due to the functions chosen as eigenfunctions of the heat conduction problem in the x-direction. Consequently, $X'_{1,mn}(x)$ and $X'_{2,mn}(x)$ satisfy the following orthogonality property (see Ref. [19] for the proof of this property):

$$\frac{\lambda_{2x,mn}}{\lambda_{1x,mn}}\int_{x=-a_1}^{0} X'_{1,mn}X'_{1,mk}\,\mathrm{d}x + \frac{\alpha_1 k_1 \lambda_{1x,mk}}{\alpha_2 k_2 \lambda_{2x,mk}}\int_{x=0}^{a_2} X'_{2,mn}X'_{2,mk}\,\mathrm{d}x = \begin{cases} 0 & \text{for } n \neq k \\ N_{x,mn} & \text{for } n = k \end{cases} \tag{38}$$

where the x-norm $N_{x,mn}$ may be evaluated as (see Ref. [19])

$$N_{x,mn} = \frac{\lambda_{2x,mn}}{\lambda_{1x,mn}}\left(\frac{1+\Pi^2_{1x,mn}}{2}\right) \times \left(a_1 + \frac{1}{\lambda^2_{1x,mn}k_1/h_1 + h_1/k_1}\right) + \frac{\alpha_1 k_1 \lambda_{1x,mn}}{\alpha_2 k_2 \lambda_{2x,mn}} \times \left(\frac{1+\Pi^2_{2x,mn}}{2}\right)\left(a_2 + \frac{1}{\lambda^2_{2x,mn}k_2/h_2 + h_2/k_2}\right) + \frac{\Pi_{1x,mn}}{2\lambda_{2x,mn}}\left(\frac{\alpha_1}{\alpha_2} - \frac{\lambda^2_{2x,mn}}{\lambda^2_{1x,mn}}\right) \tag{39}$$

Even though the choice of the functions (36) and (37) as x-eigenfunctions of the problem has required to develop a new orthogonality property, namely Eq. (38), such a choice has been done with the target to identify and stress those algebraic terms which take into account the heat conduction in the y-direction and affect the thermal field in the x-direction where the slab is two-layered. As an example, the last term on the right-hand side of Eq. (39) accounts exactly for the heat transfer by conduction in the y-direction in the two-directional two-region domain of Fig. 1 (another similar term appears in Eq. (61)). In fact, this term vanishes in the case of 1-D temperature field which takes place in 1-D two-layer slab since $\lambda_y = 0$ in Eq. (31). Instead, as may be noted through Eqs. (32) and (33), the heat conduction in the y-direction is completely independent of what happens across the layers (i.e., in the x-direction).

As a matter of fact, a one-dimensional thermal field can also take place in a two-dimensional two-layer domain (see Sections 9.1 and 9.2). Additionally, the last term on the right-hand side of Eq. (39) can also vanish in the case of two-dimensional temperature field. However, this occurs only when $\alpha_1 = \alpha_2$ with $k_1 \neq k_2$ (see Eq. (31) with $\lambda_y \neq 0$). In this case, the heat conduction in the y-direction does not affect the heat flow in the x-direction. On the contrary, when α_1 is very different from α_2, the term mentioned above will be very large in value and the heat exchange in the y-direction will strongly affect the temperature field along the x-direction. Therefore, a basic role is played by the thermal diffusivity of each of the two layers!

8. General solution in a double-series form

Following what was said in the previous Sections 5–7, the time-variable function $G_i(t)$ $(i = 1, 2)$ defined by Eq. (15) becomes

$$G_{i,mn}(t) = \mathrm{e}^{-(\lambda^2_{1x,mn}+\lambda^2_{y,m})\alpha_1 t} \quad t \geqslant 0 \tag{40}$$

Therefore, bearing in mind Eqs. (8), (32), (34), (35) and (40), and setting $B_{1x,mn}B_{y,m} = c_{mn}$, we can write:

$$\theta_{1,mn}(x,y,t) = c_{mn}X'_{1,mn}(x)Y'_m(y)\mathrm{e}^{-(\lambda^2_{1x,mn}+\lambda^2_{y,m})\alpha_1 t} \tag{41}$$

$$\theta_{2,mn}(x,y,t) = c_{mn}\frac{k_1\lambda_{1x,mn}}{k_2\lambda_{2x,mn}}X'_{2,mn}(x)Y'_m(y)\mathrm{e}^{-(\lambda^2_{1x,mn}+\lambda^2_{y,m})\alpha_1 t} \tag{42}$$

Then the general solution for the thermal fields $\theta_1(x,y,t)$ and $\theta_2(x,y,t)$ may be constructed by taking a double linear sum of all individual solutions given by Eqs. (41) and (42) over all y- and x-eigenvalues concerning the first layer, i.e. $\lambda_{y,m}$ and $\lambda_{1x,mn}$, and the second layer, i.e. $\lambda_{y,m}$ and $\lambda_{2x,mn}$, respectively. Thus, we have the infinite double-series shown below

$$\theta_1(x,y,t) = \sum_{m=0,1}^{\infty} Y'_m(y)\mathrm{e}^{-\lambda^2_{y,m}\alpha_1 t} \times \left[\sum_{n=0,1}^{\infty} c_{mn}X'_{1,mn}(x)\mathrm{e}^{-\lambda^2_{1x,mn}\alpha_1 t}\right] \quad (-a_1 \leqslant x \leqslant 0;\ 0 \leqslant y \leqslant b;\ t \geqslant 0) \tag{43}$$

$$\theta_2(x,y,t)=\frac{k_1}{k_2}\sum_{m=0,1}^{\infty}Y'_m(y)\mathrm{e}^{-\lambda^2_{y,m}\alpha_1 t}\times\left[\sum_{n=0,1}^{\infty}c_{mn}\frac{\lambda_{1x,mn}}{\lambda_{2x,mn}}X'_{2,mn}(x)\mathrm{e}^{-\lambda^2_{1x,mn}\alpha_1 t}\right]$$
$$(0\leqslant x\leqslant a_2;\ 0\leqslant y\leqslant b;\ t\geqslant 0)\quad(44)$$

which represent the general solution to the heat conduction problem here under consideration.

9. Temperature solution

Eqs. (43) and (44) still have to fit the initial condition (7). Therefore, they must hold that

$$F_1(x,y)=\sum_{m=0,1}^{\infty}Y'_m(y)\left[\sum_{n=0,1}^{\infty}c_{mn}X'_{1,mn}(x)\right]$$
$$(-a_1\leqslant x\leqslant 0;\ 0\leqslant y\leqslant b)\quad(45)$$

$$F_2(x,y)=\frac{k_1}{k_2}\sum_{m=0,1}^{\infty}Y'_m(y)\left[\sum_{n=0,1}^{\infty}c_{mn}\frac{\lambda_{1x,mn}}{\lambda_{2x,mn}}X'_{2,mn}(x)\right]$$
$$(0\leqslant x\leqslant a_2;\ 0\leqslant y\leqslant b)\quad(46)$$

The coefficients c_{mn} can be derived by using the y- and x-orthogonality relations, as shown in the following. Both sides of Eq. (45) are multiplied by $X'_{1,jk}(x)Y'_j(y)$, and the resulting expression is integrated over the ranges $x=-a_1$ to $x=0$ and $y=0$ to $y=b$. We obtain:

$$\int_{y=0}^{b}\int_{x=-a_1}^{0}F_1(x,y)X'_{1,jk}Y'_j\,\mathrm{d}x\,\mathrm{d}y=\sum_{m=0,1}^{\infty}\int_{y=0}^{b}Y'_mY'_j\,\mathrm{d}y\times\left[\sum_{n=0,1}^{\infty}c_{mn}\int_{x=-a_1}^{0}X'_{1,mn}X'_{1,jk}\,\mathrm{d}x\right]\quad(47)$$

Applying the y-orthogonality property (33), Eq. (47) becomes

$$\frac{1}{N_{y,m}}\int_{y=0}^{b}\int_{x=-a_1}^{0}F_1(x,y)X'_{1,mk}Y'_m\,\mathrm{d}x\,\mathrm{d}y=\sum_{n=0,1}^{\infty}c_{mn}\int_{x=-a_1}^{0}X'_{1,mn}X'_{1,mk}\,\mathrm{d}x\quad(48)$$

Similarly, both sides of Eq. (46) can be multiplied by $(\alpha_1\lambda_{1x,jk}/\alpha_2\lambda_{2x,jk})X'_{2,jk}(x)Y'_j(y)$, and the resulting expression can be integrated from $x=0$ to $x=a_2$ and from $y=0$ to $y=b$:

$$\frac{\alpha_1\lambda_{1x,jk}}{\alpha_2\lambda_{2x,jk}}\int_{y=0}^{b}\int_{x=0}^{a_2}F_2(x,y)X'_{2,jk}Y'_j\,\mathrm{d}x\,\mathrm{d}y=\frac{\alpha_1k_1\lambda_{1x,jk}}{\alpha_2k_2\lambda_{2x,jk}}\sum_{m=0,1}^{\infty}\times\int_{y=0}^{b}Y'_mY'_j\,\mathrm{d}y\left[\sum_{n=0,1}^{\infty}c_{mn}\frac{\lambda_{1x,mn}}{\lambda_{2x,mn}}\int_{x=0}^{a_2}X'_{2,mn}X'_{2,jk}\,\mathrm{d}x\right]\quad(49)$$

By using the y-orthogonality property (33), Eq. (49) reduces to

$$\frac{1}{N_{y,m}}\frac{\alpha_1\lambda_{1x,mk}}{\alpha_2\lambda_{2x,mk}}\int_{y=0}^{b}\int_{x=0}^{a_2}F_2(x,y)X'_{2,mk}Y'_m\,\mathrm{d}x\,\mathrm{d}y=\frac{\alpha_1k_1\lambda_{1x,mk}}{\alpha_2k_2\lambda_{2x,mk}}\sum_{n=0,1}^{\infty}c_{mn}\frac{\lambda_{1x,mn}}{\lambda_{2x,mn}}\int_{x=0}^{a_2}X'_{2,mn}X'_{2,mk}\,\mathrm{d}x\quad(50)$$

Summing up Eqs. (48) and (50), and applying the x-orthogonality property (38), we derive the coefficient c_{mn} with the following result

$$c_{mn}=\frac{1}{N_{x,mn}N_{y,m}}\frac{\lambda_{2x,mn}}{\lambda_{1x,mn}}\times\int_{y=0}^{b}Y'_m(y)\left[\int_{x=-a_1}^{0}F_1(x,y)X'_{1,mn}(x)\,\mathrm{d}x\right]\mathrm{d}y+\frac{1}{N_{x,mn}N_{y,m}}\frac{\alpha_1}{\alpha_2}\times\int_{y=0}^{b}Y'_m(y)\left[\int_{x=0}^{a_2}F_2(x,y)X'_{2,mn}(x)\,\mathrm{d}x\right]\mathrm{d}y\quad(51)$$

Eqs. (43) and (44), where c_{mn} may be evaluated through Eq. (51), give the desired temperature distribution (in default of the ambient temperature T_∞) as a function of position (x,y) and time t.

9.1. Temperature solution when $F_i(x,y)=F_{ix}(x)\Im_{iy}(y)$

If $F_1(x,y)=F_{1x}(x)\Im_{1y}(y)$, $F_2(x,y)=F_{2x}(x)\Im_{2y}(y)$ and $\Im_{1y}(y)=\Im_{2y}(y)=\Im_y(y)$, where $\Im_y(y)$ is a non-dimensional y-space-variable function, then c_{mn} given by Eq. (51) may be rewritten as $c_{mn}=c_{y,m}c_{x,mn}$, where

$$c_{y,m}=\frac{1}{N_{y,m}}\int_{y=0}^{b}\Im_y(y)Y'_m(y)\,\mathrm{d}y\quad(52)$$

$$c_{x,mn}=\frac{1}{N_{x,mn}}\left[\frac{\lambda_{2x,mn}}{\lambda_{1x,mn}}\int_{x=-a_1}^{0}F_{1x}(x)X'_{1,mn}(x)\,\mathrm{d}x+\frac{\alpha_1}{\alpha_2}\int_{x=0}^{a_2}F_{2x}(x)X'_{2,mn}(x)\,\mathrm{d}x\right]\quad(53)$$

Consequently, Eqs. (43) and (44) corresponding to the first and second layers, respectively, become

$$\theta_1(x,y,t) = \sum_{m=0,1}^{\infty} \vartheta_{y,m}(y,t)\left[\sum_{n=0,1}^{\infty} \theta_{1x,mn}(x,t)\right]$$
$$(-a_1 \leqslant x \leqslant 0;\ 0 \leqslant y \leqslant b;\ t \geqslant 0) \quad (54)$$

$$\theta_2(x,y,t) = \sum_{m=0,1}^{\infty} \vartheta_{y,m}(y,t)\left[\sum_{n=0,1}^{\infty} \theta_{2x,mn}(x,t)\right]$$
$$(0 \leqslant x \leqslant a_2;\ 0 \leqslant y \leqslant b;\ t \geqslant 0) \quad (55)$$

where $\theta_{1x,mn}(x,t)$, $\theta_{2x,mn}(x,t)$ and $\vartheta_{y,m}(y,t)$ are given, respectively, by

$$\theta_{1x,mn}(x,t) = c_{x,mn} X'_{1,mn}(x) \mathrm{e}^{-\lambda^2_{1x,mn}\alpha_1 t} \quad (-a_1 \leqslant x \leqslant 0;\ t \geqslant 0) \quad (56)$$

$$\theta_{2x,mn}(x,t) = c_{x,mn} \frac{k_1 \lambda_{1x,mn}}{k_2 \lambda_{2x,mn}} X'_{2,mn}(x) \mathrm{e}^{-\lambda^2_{1x,mn}\alpha_1 t}$$
$$(0 \leqslant x \leqslant a_2;\ t \geqslant 0) \quad (57)$$

$$\vartheta_{y,m}(y,t) = c_{y,m} \cdot Y'_m(y) \mathrm{e}^{-\lambda^2_{y,m}\alpha_1 t} \quad (0 \leqslant y \leqslant b;\ t \geqslant 0) \quad (58)$$

Bearing in mind the expressions of $Y'_m(y)$ and $N_{y,m}$ given in Table 2, it may be noted that c_m defined by Eq. (52) is a dimensionless coefficient. Consequently, $\vartheta_{y,m}(y,t)$ given by Eq. (58) is also a non-dimensional function.

If the two-layer composite slab is initially characterised by a temperature distribution dependent only on the space coordinate x, i.e. $F_1(x,y) = F_{1x}(x)$ and $F_2(x,y) = F_{2x}(x)$ in Eq. (51) (which means $\Im_y(y) = 1$ in Eq. (52)), then the expression (52) for $c_{y,m}$ simplifies to the one given in Table 2 (the expression (53) for $c_{x,mn}$ does not change, of course!). In this particular case, if the boundary surfaces in the y-direction are kept insulated (i.e., the y-boundary conditions are of the second kind and homogeneous ⇒ "*case 2*" in Table 2), then there is no temperature variation in the y-direction. In fact, bearing in mind Eqs. (30) and (31) combined to case 2 of Table 2, Eqs. (54) and (55) reduce, respectively, to

$$\theta_1(x,t) = \sum_{n=0,1}^{\infty} \theta_{1x,n}(x,t) \quad (-a_1 \leqslant x \leqslant 0;\ t \geqslant 0) \quad (59)$$

$$\theta_2(x,t) = \sum_{n=0,1}^{\infty} \theta_{2x,n}(x,t) \quad (0 \leqslant x \leqslant a_2;\ t \geqslant 0) \quad (60)$$

where the subscript '$m = 0$' does not appear since the x-eigenvalues $\lambda_{1x,n}$ $(n = 0,1,2,3,\ldots)$ are independent of the only '*efficient*' y-eigenvalue, i.e. $\lambda_{y,m=0} = 0$ (see Table 2). The results expressed by Eqs. (59) and (60) are in accordance with the ones reached by Salt [3,4], Mikhailov and Özişik [5], and Beck [23]. Therefore, 'to have a one-dimensional thermal field in the x-direction, we would have homogeneous boundary conditions of the second kind in the y-direction and an initial temperature distribution dependent only on the space coordinate x'.

9.2. Temperature solution when $F_i(x,y) = \theta_0$

If the two-dimensional two-layer composite slab is initially at a uniform temperature, i.e. $F_1(x,y) = F_2(x,y) = \theta_0$ in Eq. (51) (which means $\Im_y(y) = 1$ in Eq. (52) and $F_{1x}(x) = F_{2x}(x) = \theta_0$ in Eq. (53)), then the expression for $c_{y,m}$ does not change and is still given in Table 2, while the expression (53) for $c_{x,mn}$ becomes

$$c_{x,mn} = \frac{\theta_0}{N_{x,mn}\lambda_{2x,mn}} \left\{ \left[\frac{\lambda^2_{2x,mn}}{\lambda^2_{1x,mn}} \cos(\lambda_{1x,mn} a_1) - \frac{\alpha_1}{\alpha_2} \cos(\lambda_{2x,mn} a_2) \right] + \left[\frac{\lambda^2_{2x,mn}}{\lambda^2_{1x,mn}} \Pi_{1x,mn} \sin(\lambda_{1x,mn} a_1) - \frac{\alpha_1}{\alpha_2} \Pi_{2x,mn} \sin(\lambda_{2x,mn} a_2) \right] + \left(\frac{\alpha_1}{\alpha_2} - \frac{\lambda^2_{2x,mn}}{\lambda^2_{1x,mn}} \right) \right\} \quad (61)$$

Of course, in the case here under consideration of 'uniform initial temperature distribution, homogeneous y-boundary conditions of the second kind make one-dimensional the temperature field in the x-direction'.

Similarly to what was said for Eq. (39), the last term between round brackets on the right-hand side of Eq. (61) accounts for the heat conduction in the y-direction and affect the thermal field in the x-direction. In fact, this term vanishes in the case of 1-D temperature field which takes place in 1-D two-layer domain since $\lambda_y = 0$ in Eq. (31). Concerning this, the coefficient $c_{x,mn}$ provided by Eq. (61) reduces to the form for one-directional two-layer planar geometry [17]. It should be noted that the term mentioned above can also vanish in the case of two-dimensional temperature field. However, this happens only when $\alpha_1 = \alpha_2$ with $k_1 \neq k_2$ (see Eq. (31) with $\lambda_y \neq 0$).

10. Temperature solution when $\alpha_1 = \alpha_2$ ($k_1 \neq k_2$)

In this case, bearing in mind Eq. (31), we have that $\lambda_{1x} = \lambda_{2x} = \lambda_x$. Therefore, the x-eigencondition (30) simplifies to

$$\Pi_{1x}(\lambda_x) = -\frac{k_1}{k_2} \Pi_{2x}(\lambda_x) \quad (62)$$

where $\Pi_{1x}(\lambda_x)$ and $\Pi_{2x}(\lambda_x)$ are derived from Eq. (17) simply setting $\lambda_{1x} = \lambda_{2x} = \lambda_x$. The roots $\lambda_{x,n}$ $(n = 0,1,2,3,\ldots)$ of the transcendental equation (62) are the x-eigenvalues related to both the first and second layers and are independent of $\lambda_{y,m}$ $(m = 0,1,2,3,\ldots)$ given in Table 2. Therefore, the x-eigenfunctions $X'_{1,n}(x)$ and $X'_{2,n}(x)$ corresponding to the x-eigenvalues $\lambda_{x,n}$ may be defined as (see Eqs. (36) and (37)):

$$X'_{1,n}(x) = \sin(\lambda_{x,n} x) + \Pi_{1x,n} \cos(\lambda_{x,n} x) \quad x \in [-a_1, 0] \quad (63)$$

$$X'_{2,n}(x) = \sin(\lambda_{x,n}x) + (k_2/k_1)\Pi_{1x,n}\cos(\lambda_{x,n}x) \quad x \in [0, a_2] \tag{64}$$

and satisfy the following orthogonality property from Eq. (38)

$$\int_{x=-a_1}^{0} X'_{1,n}X'_{1,k}\,dx + \frac{k_1}{k_2}\int_{x=0}^{a_2} X'_{1,n}X'_{1,k}\,dx = \begin{cases} 0 & \text{for } n \neq k \\ N_{x,n} & \text{for } n = k \end{cases} \tag{65}$$

The x-norm $N_{x,n}$ may be evaluated from Eq. (39) as

$$N_{x,n} = \frac{1+\Pi_{1x,n}^2}{2}\left(a_1 + \frac{1}{\lambda_{x,n}^2 k_1/h_1 + h_1/k_1}\right) + \frac{k_1}{k_2}\left(\frac{1+\Pi_{2x,n}^2}{2}\right)\left(a_2 + \frac{1}{\lambda_{x,n}^2 k_2/h_2 + h_2/k_2}\right) \tag{66}$$

From what was said previously, it follows that solution to the problem (when $\alpha_1 = \alpha_2$ with $k_1 \neq k_2$) may be searched splitting up the corresponding 2-D eigenvalue problem in two, separated, 1-D eigenvalue problems, one across the layers (i.e., in the x-direction) and the other along the composite slab (i.e., in the y-direction). Both the eigenproblems are special cases of the classical Sturm–Liouville problem [6,7].

Then the general temperature solution represented by Eqs. (43) and (44) for the first and second layers, respectively, becomes

$$\theta_1(x,y,t) = \sum_{m=0,1}^{\infty} Y'_m(y)e^{-\lambda_{y,m}^2\alpha t}\left[\sum_{n=1,2}^{\infty} c_{mn}X'_{1,n}(x)e^{-\lambda_{x,n}^2\alpha t}\right] \quad (-a_1 \leqslant x \leqslant 0;\ 0 \leqslant y \leqslant b;\ t \geqslant 0) \tag{67}$$

$$\theta_2(x,y,t) = \frac{k_1}{k_2}\sum_{m=0,1}^{\infty} Y'_m(y)e^{-\lambda_{y,m}^2\alpha t}\left[\sum_{n=1,2}^{\infty} c_{mn}X'_{2,n}(x)e^{-\lambda_{x,n}^2\alpha t}\right] \quad (0 \leqslant x \leqslant a_2;\ 0 \leqslant y \leqslant b;\ t \geqslant 0) \tag{68}$$

where $\lambda_{x,0} = 0$ is a root of the x-eigencondition (62) but it is not an efficient eigenvalue of the problem since corresponding to $X'_{1,0} = 0$ and $X'_{2,0} = 0 \Rightarrow n = 1, 2, 3, \ldots$ in Eqs. (67) and (68). The coefficient c_{mn} appearing in the above equations may be evaluated from Eq. (51) as

$$c_{mn} = \frac{1}{N_{x,n}N_{y,m}}\int_{y=0}^{b} Y'_m(y)\left[\int_{x=-a_1}^{0} F_1(x,y)X'_{1,n}(x)\,dx\right]dy + \frac{1}{N_{x,n}N_{y,m}}\int_{y=0}^{b} Y'_m(y)\left[\int_{x=0}^{a_2} F_2(x,y)X'_{2,n}(x)\,dx\right]dy \tag{69}$$

If $F_1(x,y) = F_{1x}(x)\Im_{1y}(y)$, $F_2(x,y) = F_{2x}(x)\Im_{2y}(y)$ and $\Im_{1y}(y) = \Im_{2y}(y) = \Im_y(y)$, where $\Im_y(y)$ is a non-dimensional space-variable function, then c_{mn} given by Eq. (69) may be rewritten as $c_{mn} = c_{y,m}c_{x,n}$, where

$$c_{y,m} = \frac{1}{N_{y,m}}\int_{y=0}^{b} \Im_y(y)Y'_m(y)\,dy \tag{70}$$

$$c_{x,n} = \frac{1}{N_{x,n}}\left[\int_{x=-a_1}^{0} F_{1x}(x)X'_{1,n}(x)\,dx + \int_{x=0}^{a_2} F_{2x}(x)X'_{2,n}(x)\,dx\right] \tag{71}$$

Consequently, Eqs. (67) and (68) may be rewritten, respectively, as

$$\theta_1(x,y,t) = \theta_{1x}(x,t)\vartheta_y(y,t) \tag{72}$$

$$\theta_2(x,y,t) = \theta_{2x}(x,t)\vartheta_y(y,t) \tag{73}$$

where $\theta_{1x}(x,t)$, $\theta_{2x}(x,t)$ and $\vartheta_y(y,t)$ are

$$\theta_{1x}(x,t) = \sum_{n=1,2}^{\infty} c_{x,n}X'_{1,n}(x)e^{-\lambda_{x,n}^2\alpha t} \quad (-a_1 \leqslant x \leqslant 0;\ t \geqslant 0) \tag{74}$$

$$\theta_{2x}(x,t) = \frac{k_1}{k_2}\sum_{n=1,2}^{\infty} c_{x,n}X'_{2,n}(x)e^{-\lambda_{x,n}^2\alpha t} \quad (0 \leqslant x \leqslant a_2;\ t \geqslant 0) \tag{75}$$

$$\vartheta_y(y,t) = \sum_{m=0,1}^{\infty} c_{y,m}Y'_m(y)e^{-\lambda_{y,m}^2\alpha t} \quad (0 \leqslant y \leqslant b;\ t \geqslant 0) \tag{76}$$

Bearing in mind the expressions of $Y'_m(y)$ and $N_{y,m}$ given in Table 2, it follows that $c_{y,m}$ defined by Eq. (70) is a dimensionless coefficient. Consequently, $\vartheta_y(y,t)$ given by Eq. (76) is also a non-dimensional function.

The results expressed through Eqs. (72) and (73) state that the solution of a two-dimensional two-layer homogeneous boundary-value problem of unsteady heat conduction may be written readily as the product of the solutions of two, separated, one-dimensional problems, one along the x-direction (two regions) and the other along the y-direction (single region). This may be done if the initial temperature distribution in each layer of the body is given as a product of single space-variable functions, i.e. $F_1(x,y) = F_{1x}(x)\Im_{1y}(y)$ and $F_2(x,y) = F_{2x}(x)\Im_{2y}(y)$, and if the single space-variable functions along the direction characterised by only one region are independent of the layer, i.e. $\Im_{1y}(y) = \Im_{2y}(y) = \Im_y(y)$. Therefore, the '*product solution*' technique may be applied not only to multi-dimensional single-layer heat conduction problems [21,22] but also to multi-dimensional multi-layer ones.

Obviously, the case of initial temperature distribution dependent only on the space coordinate x (i.e., $F_1(x,y) = F_{1x}(x)$ and $F_2(x,y) = F_{2x}(x)$ in Eq. (69) which means $\Im_y(y) = 1$ in Eq. (70)), is also expressible in the product form. In this case, the expression (70) for $c_{y,m}$ simplifies to the one given in Table 2 (the expression for $c_{x,n}$ does not change, of course!).

Similarly, the case of uniform initial temperature distribution (i.e., $F_1(x,y) = F_2(x,y) = \theta_0$ in Eq. (69), which means $\Im_y(y) = 1$ in Eq. (70) and $F_{1x}(x) = F_{2x}(x) = \theta_0$ in Eq. (71)), is also expressible in the product form. In this case, the expression for $c_{y,m}$ does not change and is still given in Table 2, while the expression (71) for $c_{x,n}$ becomes

$$c_{x,n} = \frac{\theta_0}{N_{x,n}\lambda_{x,n}}[\cos(\lambda_{x,n}a_1) - \cos(\lambda_{x,n}a_2) + \Pi_{1x,n}\sin(\lambda_{x,n}a_1) - \Pi_{2x,n}\sin(\lambda_{x,n}a_2)] \tag{77}$$

11. Computation of the transverse eigenvalues

Introducing the dimensionless variables β_ξ, Bi_1, Bi_2, κ and γ defined in nomenclature, the x-eigencondition (62) becomes

$$\Pi_{1\xi}(\beta_\xi) + \frac{1}{\kappa}\Pi_{2\xi}(\beta_\xi) = 0 \tag{78}$$

where the functions $\Pi_{1\xi}(\beta_\xi)$ and $\Pi_{2\xi}(\beta_\xi)$ are (see Eq. (17) with $\lambda_{1x} = \lambda_{2x} = \lambda_x$)

$$\Pi_{1\xi}(\beta_\xi) = \frac{\beta_\xi + Bi_1\tan(\beta_\xi)}{Bi_1 - \beta_\xi\tan(\beta_\xi)}$$
$$\Pi_{2\xi}(\beta_\xi) = \frac{\kappa\beta_\xi + Bi_2\tan(\gamma\beta_\xi)}{Bi_2 - \kappa\beta_\xi\tan(\gamma\beta_\xi)} \tag{79}$$

It may be proven [19] that the roots $\beta_{\xi,n}$ $(n = 1,2,3,\ldots)$ of the transcendental equation (78) are all real (the negative roots are equal in absolute value to the positive ones) and also form a monotonically increasing infinite series according to the classical Sturm–Liouville problem.

Several algorithms may be used for computing transverse eigenvalues of Eq. (78) and are available in the specialised literature [9–11]. These algorithms usually require the user to supply two points such that the function values at these two points have opposite sign. For equations similar to the (78), where it is difficult to obtain two such points, Aviles-Ramos et al. [11] developed a procedure to find the region where the root is located. Then, a high-order Newton's method was suggested by the same authors to compute the root. Alternatively, an algorithm based on Müller's method [20] may be successfully and efficiently used to find the roots of the eigencondition (78). It requires the user to supply a vector $\boldsymbol{\zeta}_\xi$ of length p containing the initial guesses $\zeta_{\xi,n}$ for the dimensionless transverse eigenvalues $\beta_{\xi,n}$ $(n = 1,2,3,\ldots,p)$, which are the components of the eigenvector $\boldsymbol{\beta}_\xi$. The number p may readily be established through a criterion derived in [17] and used in the numerical example of Section 12. It should be noted that the 'building' of the vector $\boldsymbol{\zeta}_\xi$ for Eq. (78) is the starting step and the most difficult step for reaching convergence (i.e., $\boldsymbol{\beta}_\xi$ of Eq. (78)) of Müller's method accurately and rapidly.

Then, as shown in Ref. [19] where the treatment is completely and exhaustively developed, the initial guesses $\zeta_{\xi,n}$ may be evaluated as

$$\zeta_{\xi,n} = \frac{\beta_{\xi,\mathrm{h},n}(Bi_{\mathrm{h},1}, Bi_{\mathrm{h},2})}{(1+\gamma)} \tag{80}$$

where $\beta_{\xi,\mathrm{h},n}$ represents the nth dimensionless transverse eigenvalue of a homogeneous (single-layer) rectangular domain equivalent to the two-dimensional two-layered domain here under consideration (Fig. 1). This equivalence concerns both the geometry (same dimensions, i.e. a_1, a_2 and b, and same origin for the $0xy$ frame of reference) and the boundary conditions (of third kind, with the same heat transfer coefficients, i.e. h_1 and h_2). The eigenvalues $\beta_{\xi,\mathrm{h},n}$ of the homogeneous domain are the roots of the following well-known transcendental equation:

$$\tan(\beta_{\xi,\mathrm{h}}) = \frac{(Bi_{\mathrm{h},1} + Bi_{\mathrm{h},2})\beta_{\xi,\mathrm{h}}}{\beta_{\xi,\mathrm{h}}^2 - Bi_{\mathrm{h},1}Bi_{\mathrm{h},2}} \tag{81}$$

where $Bi_{\mathrm{h},i} = Bi_i(1+\gamma)k_1/k_\mathrm{h}$ and k_h may be evaluated as [19]

$$\frac{k_1}{k_\mathrm{h}} = \frac{\kappa + \gamma}{\kappa \cdot (\gamma + 1)} \tag{82}$$

Therefore, $\zeta_{\xi,n}$ defined by Eq. (80) depend on Bi_1, Bi_2, κ and γ, which are the *sole* groups characterising $\beta_{\xi,n}$ (see Eq. (78)). The eigenvalues $\beta_{\xi,\mathrm{h},n}$ of Eq. (81) may be readily and easily calculated by using the explicit approximate relations (with, at least, six-decimal place accuracy) defined by Haji-Sheikh and Beck [24] for homogeneous bodies with convective boundary conditions, followed by whatever root-finding iteration to realise a high degree of accuracy. Once the initial guesses $\zeta_{\xi,n}$ have been established, the convergence (i.e., $\beta_{\xi,n}$ of Eq. (78)) of Müller's method may be reached only if the transverse eigencondition (78) is given in the following form [19] (without vertical asymptotes):

$$\begin{aligned}&[Bi_1Bi_2\sin(\beta_\xi)\cos(\gamma\beta_\xi) - \kappa\beta_\xi^2\sin(\gamma\beta_\xi)\cos(\beta_\xi)]\\ &\quad + g_1(Bi_1,Bi_2,\kappa,\beta_\xi)[\sin(\gamma\beta_\xi)\cos(\beta_\xi) - \sin(\beta_\xi)\cos(\gamma\beta_\xi)]\\ &\quad + \beta_\xi g_2(Bi_1,Bi_2,\kappa)[1 - g_3(Bi_1,Bi_2,\kappa)\sin(\gamma\beta_\xi)\sin(\beta_\xi)]\\ &= 0\end{aligned} \tag{83}$$

where the quantities g_i $(i = 1,2,3)$ are given in [19] (however, it is a simple matter to derive them). The procedure here proposed to estimate the initial guesses $\zeta_{\xi,n}$ (required by the algorithm based on Müller's method) for the dimensionless transverse eigenvalues

$\beta_{\xi,n}$ of Eq. (83) has been carefully verified for wide ranges of Bi_1, Bi_2, κ and γ, and for several their combinations. The convergence is always reached accurately and in a short time. Of course, $\beta_{\xi,n} \equiv \zeta_{\xi,n}$ only when $\kappa = 1$.

12. Numerical example

A two-layer rectangular region ($-a_1 \leqslant x \leqslant a_2$, $0 \leqslant y \leqslant b$; see Fig. 1) is initially at a uniform temperature T_0. For times $t > 0$, the boundaries at $x = -a_1$ and $x = a_2$ dissipate heat by convection into an environment at temperature $T_\infty \neq T_0$. Instead, the boundaries at $y = 0$ and $y = b$ are subjected to a jump in the temperature and are kept at $T_\infty \neq T_0$ for $t > 0$ ($\Rightarrow$ case 1 in Tables 1 and 2). For the dimensionless variables characterising the problem, the following values are fixed: $\gamma = 2$, $\mu = 4$, $Bi_1 = 1$, $Bi_2 = 2$ and $\kappa = 4$.

The ξ-eigencondition (83) may numerically be solved for the determination of the dimensionless transverse eigenvalues $\beta_{\xi,n}$ as illustrated in the previous section. In particular, the number p of transverse eigenvalues which has to be used in the dimensionless series solutions $\Theta_{1\xi} = \theta_{1x}/\theta_0$ and $\Theta_{2\xi} = \theta_{2x}/\theta_0$, where θ_{1x} and θ_{2x} are defined by Eqs. (74) and (75), respectively, has been established by requiring that the exact ($p = n = \infty$) and approximate ($p =$ finite) solutions in the ξ-direction differ by not more than 3.5% in absolute value. Of course, the maximum deviation between the exact and approximate non-dimensional temperatures along ξ is obtained for $\tau = 0$ and in correspondence to the ξ-boundary characterised by the lower value of the Biot number, i.e. at $\xi = -1$. It may be proven that, when $p = 20$, the percent deviation in absolute value for $\tau = 0$ is less than 3.2% at $\xi = -1$. Instead, for $\tau = 0$, it is less than 1.5% at $\xi = \gamma = 2$. The first 20 dimensionless ξ-eigenvalues $\beta_{\xi,n}$ of the eigencondition in the form (83) are given in Table 3, together with the initial guesses $\zeta_{\xi,n}$. For sake of completeness, the number of iterations N_{IT} for the convergence is also given as well as the results for the eigencondition in the form (78). A good four eigenvalues are filtered by itself!

Fig. 2 shows the dimensionless isothermal curves $\Theta_i(\xi, \psi, \tau) = \text{const}$ within the two-dimensional two-layered slab during the transient heat transfer between slab and surrounding fluid at four different dimensionless times. It may be noted that the isothermal curves (normal to the heat flux lines at any point) have a kink at the interface of the two-layer rectangular domain. In particular, according to Eq. (6), the temperature gradient is larger in the first layer which is characterised by the lower thermal conductivity ($\kappa = 4$). Additionally, in view of the values prescribed for Bi_1, Bi_2 and κ, the isothermal curves thicken within the first slab-shaped region in the ξ-coordinate direction.

Table 3
Eigenvector $\boldsymbol{\beta}_\xi$ (including first 20 transverse eigenvalues) obtained starting from the initial guess vector $\boldsymbol{\zeta}_\xi$ defined by Eq. (80) when $Bi_1 = 1$, $Bi_2 = 2$, $\gamma = 2$ and $\kappa = 4$

n	$\boldsymbol{\zeta}_\xi$	Eq. (83)		Eq. (78)	
		$\boldsymbol{\beta}_\xi$	N_{IT}	$\boldsymbol{\beta}_\xi$	N_{IT}
1	0.526109717999275	0.488378138974359	7	0.488378138974359	9
2	1.312935899054485	1.549657105379747	8	1.549657105379747	10
3	2.256266866974628	2.170835074756995	7	2.170835074756995	8
4	3.255303632400386	3.243084139021714	7	3.243084139021714	12
5	4.275829090717518	4.511710306627925	8	4.511710306627925	10
6	5.306309217806367	5.151228190274521	8	5.151228190274521	10
7	6.342108755141915	6.335621173166092	7	–	–
8	7.381058180290705	7.606195860004210	9	7.606195860004210	13
9	8.422018658121692	8.248172858352781	9	8.248172858352781	14
10	9.464338390980981	9.459960892229189	7	–	–
11	10.507618574944000	10.726875700234650	10	10.726875700234650	16
12	11.551601961050590	11.369585068212270	10	11.369585068212270	14
13	12.596115362377250	12.592818102294480	7	–	–
14	13.641038001116270	13.856732824312450	11	13.856732824312450	23
15	14.686283123158100	14.499751333696300	10	14.499751333696300	15
16	15.731786829280720	15.729143733192350	7	15.729143733192350	11
17	16.777501027096940	16.990828456225510	11	16.990828456225510	16
18	17.823388835589520	17.634005247721890	10	17.634005247721890	24
19	18.869421502533970	18.867216516684070	7	–	–
20	19.915576284699860	20.127221886762410	11	20.127221886762410	16

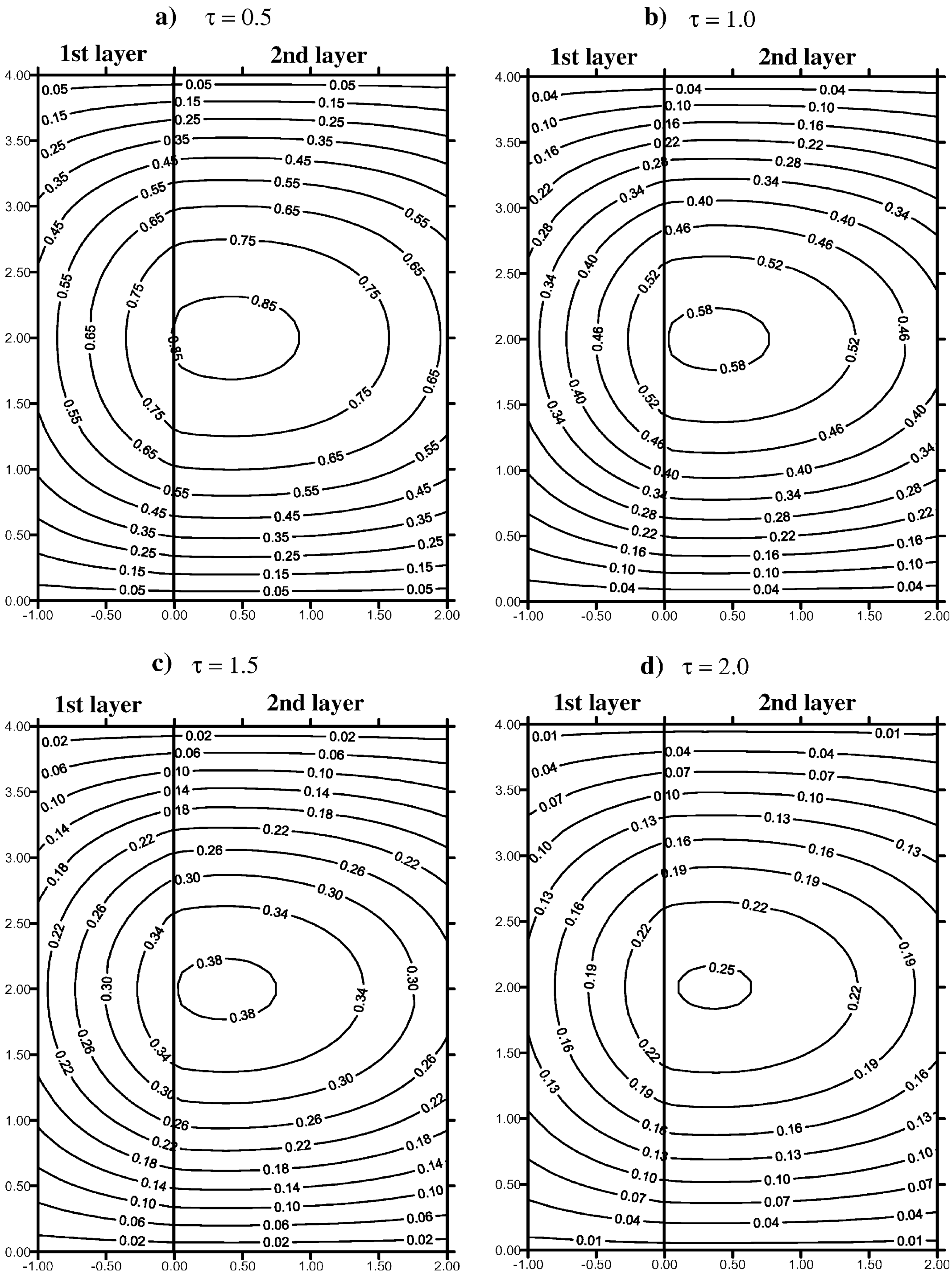

Fig. 2. Isothermal curves for a two-dimensional two-layer slab, characterised by $\xi \in [-1, \gamma = 2]$ and $\psi \in [0, \mu = 4]$, at different dimensionless times: (a) $\tau = 0.5$; (b) $\tau = 1.0$; (c) $\tau = 1.5$; and (d) $\tau = 2.0$.

13. Conclusions

Heat conduction in two-directional two slab-shaped regions has been analytically investigated in a time-dependent condition. It has been shown that plane temperature fields, which appear in geometric two-dimensional heat flow, are significantly more difficult to calculate than the corresponding cases in which temperature only changes in one coordinate direction perpendicular to the layers. In particular, it has been shown that the fulfilment of the inner boundary conditions is possible only if constant temperature kept at zero or adiabatic edges are rigorously prescribed at the outer boundary conditions along the layers. Furthermore, homogeneous boundary conditions of the second kind along the layers and a either uniform or variable across the layers initial temperature distribution make one-dimensional the temperature field in the direction perpendicular to the layers. Therefore, in this case, the composite slab may be considered as a '*lumped* (*thermal*) *capacitance system*' in the direction parallel to the layers.

The transverse eigenvalues have been computed by the use of an efficient and accurate procedure based on the concept of homogeneous (single-layer) rectangular domain equivalent to the two-dimensional two-layered domain under consideration. This concept has been used for searching the initial guesses for the transverse eigenvalues inherent to the eigenproblem associated to the transient multi-layer heat conduction. Then, the Müller root-finding iteration has been utilised to compute the eigenvalues.

By setting thermal diffusivity ratio unitary, a simplified solution for transient thermal response (dealing properly with thermal conductivity ratios of all magnitudes) emerges. In fact, it can be expressed as the product of two, separated, one-directional solutions, one across the layers and the other along the composite slab, provided the initial temperature distribution in each layer of the body is either: (1) given as a product of single space-variable functions, where the single space-variable functions parallel to the layers are independent of the layer; either (2) dependent only on the space coordinate in the direction perpendicular to the layers, or (3) simply uniform.

Finally, the searched complete temperature solution prepared for computer implementation can be used to aid in the verification of approximate numerical commercial programs.

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